

Tensor Perturbations in Quantum Cosmological Backgrounds

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ABSTRACT: In the description of the dynamics of tensor perturbations on a homogeneous and isotropic background cosmological model, it is well known that a simple Hamiltonian can be obtained if one assumes that the background metric satisfies Einstein classical field equations. This makes it possible to analyze the quantum evolution of the perturbations since their dynamics depends only on this classical background. In this paper, we show that this simple Hamiltonian can also be obtained from the Einstein-Hilbert lagrangian without making use of any assumption about the dynamics of the background metric. In particular, it can be used in situations where the background metric is also quantized, hence providing a substantial simplification over the direct approach originally developed by Halliwell and Hawking.

KEYWORDS: cosmological perturbation theory – gravity waves/theory – physics of the early universe.

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1. Introduction

In the theory of the evolution of cosmological perturbations, simple equations have been obtained using the assumption that the background model satisfies classical General Relativity. Lagrangians (and Hamiltonians) describing the dynamics of scalar, vector, and tensor perturbations coming from the Einstein-Hilbert lagrangian have been greatly simplified in different cosmological scenarios under the assumption that the background metric satisfies Einstein classical field equations, after taking out space and time total derivatives [1]. In such a framework, the quantization of these perturbations becomes easy, with a quite simple interpretation: they can be seen as quantum fields which behave essentially as scalar fields with a time dependent effective mass. The time varying scale factor which is responsible for this “mass” acts

as a pump field [2], creating or destroying modes of the perturbations. Under the assumption of an initial vacuum state, the spectrum of perturbations can be obtained and compared with observations. This was done in particular in the cosmological inflationary scenario [3], with a resulting spectrum for scalar perturbations in good agreement with the data [4].

As a step forward, and as the overwhelming majority of classical backgrounds possess an initial singularity at which the classical theory is expected to break down, it is important to study the quantum evolution of the perturbations when the background is also quantized. In recent years, many such quantum background cosmological models have been proposed which share this attractive property of exhibiting neither singularities nor horizons [5, 6, 7]. However, the usual treatment for cosmological perturbations in quantum cosmological backgrounds, *i.e.* taking the Einstein-Hilbert lagrangian and expanding it up to second order without using the classical background solution, in general yields extremely complicated Hamiltonians, and, consequently, quantum equations that are difficult to handle [8].

The aim of this paper is to show that these complicated equations can be transformed into much simpler equations, at least in the case of tensor perturbations, similar to the equations present in the classical General Relativity case, without making any assumption about the background dynamics. As the matter fields model we take a general perfect fluid, for which several simple quantum cosmological solutions at zeroth order [5, 7] are known, and we briefly discuss, for the sake of completeness, the case of matter being in the form of a scalar field.

In order to achieve our goal, in the Hamiltonian point of view, we show that, without ever using the background equations, there are classical and quantum canonical transformations (see, *e.g.* Ref. [9]), which we exhibit explicitly, connecting the original Hamiltonian to the simpler one that would have been obtained had we used the background classical field equations and, consequently, a much simpler quantum mechanical functional equation. In the lagrangian point of view, we eliminate a total time derivative from the Einstein-Hilbert action, obtaining a lagrangian with a second order derivative of the scale factor. We then use Ostrogradski method [10] to handle these second derivative terms in the lagrangian. Using the theory of constrained systems to deal with the second class constraints that subsequently appear, we recover the same simple Hamiltonian obtained through canonical transformations.

The Hamiltonian can be further simplified in the case for which the background is classical through a time dependent canonical transformation where the scale factor is viewed as a given function of time. When the background is also quantized, this further step can only be implemented provided one uses an ontological interpretation of quantum mechanics, whereby quantum Bohmian trajectories at zeroth order can be obtained and given a meaning. Then, and only in this case, can the “pump field” be treated in the gravitational wave quantum equations as a given function of time.

The paper is organized as follows. In the following section 2, we specify the ac-

tion and Hamiltonian by restricting attention to the particular case of a Friedmann-Lemaître-Robertson-Walker (FLRW) background over which we concentrate on tensor perturbations only. This is done in the case of a perfect fluid, and provides a summary of the general formalism first obtained in Ref. [8]. Then, after taking out a total time derivative, we assume that the background metric satisfies the zeroth order Einstein equations in order to simplify the action, thus getting the simple Hamiltonian of Ref. [1]. The core of this paper is then presented in Sec. 3, in which we show how to obtain the simplified Hamiltonian without ever using that the background metric satisfies the zeroth order Einstein equations, in fact, without assuming any background dynamics, both in the Hamiltonian and Lagrangian point of views. In Sec. 4, we show how further simplifications can be achieved if one assumes the Bohm-de Broglie interpretation. This is also where we briefly discuss the scalar field case. Finally, Sec. 5 ends this paper with some general conclusions.

2. Review of Tensor Perturbations in a Classical Friedmann Lemaître Robertson Walker Background

In this section we review the procedures to obtain the Hamiltonian governing the classical and quantum dynamics of tensor perturbations when one assumes that the cosmological background satisfies classical Einstein's equations.

2.1 General action and Hamiltonian

Let us consider the case in which the background spacetime is of the Friedmann-Lemaître-Robertson-Walker (FLRW) type, over which we wish to investigate tensor perturbations. As it is well known, this spacetime may be foliated by space-like hypersurfaces which are maximally symmetric, and we shall also impose that these hypersurfaces be compact. Therefore, the line element may be written as

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j, \quad (2.1)$$

where t is the physical time as measured by an observer co-moving with the maximally symmetric hypersurfaces, $a(t)$ is the so called scale factor and γ_{ij} is the metric of the spacelike hypersurface.

This line element may be alternatively written as

$$ds^2 = a^2(\eta) (d\eta^2 - \gamma_{ij}dx^i dx^j), \quad (2.2)$$

where η is called the conformal time. We shall in what follows use the more general expression

$$ds^2 = N^2(\tau)d\tau^2 - a^2(\tau)\gamma_{ij}dx^i dx^j. \quad (2.3)$$

Both expressions (2.1) and (2.2) then follow from (2.3) by adequate choices of the function N and concomitantly of the timelike coordinate τ . Specifically, (2.2) is

equivalent to $N = a$. In the ADM formalism, N is the lapse function. For FLRW spacetimes the shift vector may be made identically zero.

We shall now perturb the above geometry by adding to the metric the infinitesimal quantities w_{ij} in the following manner:

$$ds^2 = N^2(\tau) d\tau^2 - a^2(\tau) (\gamma_{ij} + w_{ij}) dx^i dx^j, \quad (2.4)$$

which transform as a tensor under diffeomorphisms of the 3-surface onto itself; w_{ij} satisfy the identities

$$w^{ij}{}_{|i} = 0, \quad \text{and} \quad w^i{}_i = 0. \quad (2.5)$$

In the above expression, the bar stands for the covariant derivative using as connections the Christoffel symbols calculated from γ_{ij} , and indices of w_{ij} are raised and lowered by this metric. It is the purpose of the present article to study the dynamics of the tensor modes w_{ij} , which are the ones that give rise to gravitational waves on the FLRW background. It is also possible to show that these tensor modes are gauge invariant and therefore can never be coordinate artifacts, having a clear physical interpretation as real perturbations.

Let us now suppose that the FLRW background is filled with a fluid that obeys the equation of state

$$p = (\lambda - 1)\rho, \quad (2.6)$$

p and ρ being respectively the pressure and energy density of the fluid, and λ being a constant. Note that all the following results can easily be generalized to any matter field minimally coupled to gravity. The example of a minimally coupled scalar field is actually treated in section 4.2. Here, we chose a perfect fluid because there are some known (and simple) minisuperspace quantum solutions in the literature [5, 7] in that case, which can be immediately applied to the following results. If we restrict ourselves to the case where only the tensor modes are present, we can show, using the background FLRW metric given in (2.3), that the gravitational part of the action, namely

$$\mathcal{S}_{\text{GR}} = -\frac{1}{6\ell_{\text{Pl}}^2} \int \sqrt{-g} R d^4x, \quad (2.7)$$

$[\ell_{\text{Pl}} = (8\pi G_{\text{N}}/3)^{1/2}]$ being the Planck length] may be written as

$$\begin{aligned} \mathcal{S}_{\text{GR}} = -\frac{1}{6\ell_{\text{Pl}}^2} \int d^4x \gamma^{1/2} a^3 \bigg[& \frac{6\mathcal{H}^2}{N} - \frac{6Nk}{a^2} - \frac{\dot{w}^{ij}\dot{w}_{ij}}{4N} + \frac{Nw^{ij|k}w_{ij|k}}{4a^2} \\ & - \left(\frac{3\mathcal{H}^2}{2N} - \frac{kN}{a^2} \right) w_{ij}w^{ij} - \frac{2\mathcal{H}w^{ij}\dot{w}_{ij}}{N} \bigg], \end{aligned} \quad (2.8)$$

where $\mathcal{H} \equiv \dot{a}/a$ and the dot means derivative with respect to coordinate time τ .

For the matter part of the action, the fluid has as lagrangian density

$$\mathcal{L}_{\text{matter}} = p\sqrt{-g} \quad (2.9)$$

As the tensor modes are the only perturbations present, and as they obey Eqs. (2.5), no combination of them can appear in (2.9). Conversely, by the assumed linearity of the dynamics of the perturbations, no perturbation of p , which is a scalar, can couple to the tensor mode under study. Hence, the pressure and energy density of the fluid will be kept to their background values. Therefore, the matter part of the action changes according to

$$\delta^{(2)}\mathcal{S}_{\text{matter}} = \int d^4x \gamma^{1/2} N a^3 \left(1 - \frac{1}{4} w^{ij} w_{ij}\right) p. \quad (2.10)$$

The total action reads

$$\mathcal{S} = - \int \frac{d^4x}{6\ell_{\text{Pl}}^2} \gamma^{1/2} a^3 \left[\frac{6\mathcal{H}^2}{N} - \frac{6Nk}{a^2} - \frac{\dot{w}^{ij}\dot{w}_{ij}}{4N} + \frac{Nw^{ij|k}w_{ij|k}}{4a^2} - \left(\frac{3\mathcal{H}^2}{2N} - \frac{kN}{a^2} \right) w_{ij}w^{ij} - \frac{2\mathcal{H}w^{ij}\dot{w}_{ij}}{N} \right] + \int d^4x \gamma^{1/2} a^3 N p \left(1 + \frac{w_{ij}w^{ij}}{4}\right). \quad (2.11)$$

The pure gravitational part of this action is the one that appears in Ref. [8], where the tensorial modes are decomposed in terms of the normal tensor modes, which are eigenfunctions of the Laplacian operator on the 3-sphere (in that article, the analysis was restricted to the case $k = +1$ and the fluid is a non-minimally coupled scalar field). Of course, it generates Einstein's equations at zeroth and first order in the tensor perturbation, as can be checked.

We can now calculate the Hamiltonian coming from action (2.11). Let us first recall the expressions for the canonical momenta, namely

$$P_a = \frac{1}{6\ell_{\text{Pl}}^2} \int d^3x \gamma^{1/2} \frac{a^2}{N} (-12\mathcal{H} + 3\mathcal{H}w_{ij}w^{ij} + 2w_{ij}\dot{w}^{ij}), \quad (2.12)$$

and

$$\tilde{\Pi}^{ij} = \frac{\gamma^{1/2}a^3}{6\ell_{\text{Pl}}^2 N} \left(2\mathcal{H}w^{ij} + \frac{1}{2}\dot{w}^{ij} \right). \quad (2.13)$$

The Hamiltonian can then be calculated noting that a and P_a depend only on the parameter τ , and therefore can be taken outside the spatial integrals. In what follows, we also define the total volume of three-space through $V \equiv \int d^3x \gamma^{1/2}$, which we assume is finite, *i.e.*, as previously discussed, we consider either closed spatial sections, or compact flat or hyperbolic sections.

For the fluid part, we use the formalism of Schutz [11], in which the pressure p of the fluid is written as

$$p = p_{0r} \left(\frac{\dot{\varphi} + \theta\dot{s}}{N\lambda} \right)^{\frac{\lambda}{\lambda-1}} \exp \left[-\frac{s}{s_{0r}(\lambda-1)} \right]. \quad (2.14)$$

The variables φ , θ and s are velocity potentials for the fluid congruence with suitable thermodynamical interpretations, and p_{0r} and s_{0r} are arbitrary constants related to

the initial conditions of the fluid. The canonical momenta p_φ , p_s and p_θ can be obtained in the usual way. One can perform the canonical transformation (for details see Refs. [6, 12])

$$T = -p_s \exp\left(-\frac{s}{s_{0r}}\right) p_\varphi^{-\lambda} \rho_{0r}^{\lambda-1} s_{0r}, \quad (2.15)$$

and

$$\varphi_N = \varphi + \lambda s_0 \frac{p_s}{p_\varphi}, \quad (2.16)$$

leading to the momenta

$$p_T = \frac{p_\varphi^\lambda}{\rho_{0r}^{\lambda-1}} \exp\left(-\frac{s}{s_{0r}}\right), \quad (2.17)$$

and

$$p_{\varphi_N} = p_\varphi. \quad (2.18)$$

As it turns out, these variables are more suitable than the original ones as the fluid Hamiltonian expressed in terms of those gets a much simpler form.

We can also rescale the variables a , w , N , T , and their momenta, according to

$$\bar{a} = a \frac{\sqrt{V}}{\ell_{\text{Pl}}}, \quad (2.19)$$

$$\bar{P}_a = P_a \frac{\ell_{\text{Pl}}}{\sqrt{V}}, \quad (2.20)$$

$$\bar{w}_{ij} = \frac{w_{ij}}{\sqrt{V}}, \quad (2.21)$$

$$\bar{\Pi}^{ij} = \tilde{\Pi}^{ij} \sqrt{V}, \quad (2.22)$$

$$\bar{T} = TV, \quad (2.23)$$

$$\bar{p}_T = \frac{p_T}{V}, \quad (2.24)$$

$$\bar{N} = N \frac{\sqrt{V}}{\ell_{\text{Pl}}}, \quad (2.25)$$

and a dimensionless Hamiltonian

$$\bar{H} = H \frac{\ell_{\text{Pl}}}{\sqrt{V}}. \quad (2.26)$$

After all these procedures, the total Hamiltonian is finally put in the form (omitting from now on the bars):

$$\begin{aligned} H &\equiv NH_0 \\ &= N \left\{ -\frac{P_a^2}{4a} - ka + \frac{P_T}{a^{3(\lambda-1)}} \left[1 + \frac{(\lambda-1)}{4} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{5P_a^2}{48a} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \\
& + \int d^3x \left[\frac{6P_{(w)ij} \tilde{\Pi}^{ij}}{a^3 \gamma^{1/2}} + 2 \frac{P_a w_{ij} \tilde{\Pi}^{ij}}{a^2} + \gamma^{1/2} a \left(\frac{w^{ij|k} w_{ij|k}}{24} + \frac{k}{6} w_{ij} w^{ij} \right) \right] \Bigg\}, \quad (2.27)
\end{aligned}$$

which is nothing but the Hamiltonian of Ref. [8] expressed for a perfect fluid. This Hamiltonian, which is zero due to the constraint $H_0 \approx 0$, yields the correct Einstein equations both at zeroth and first order in the perturbations, as can be checked explicitly. Note that in order to obtain its expression, no assumption has been made about the background dynamics.

The first order equation for the tensor perturbation reads

$$\ddot{w}_{ij} - \frac{\dot{N}}{N} \dot{w}_{ij} + 3 \frac{\dot{a}}{a} \dot{w}_{ij} - \frac{N^2}{a^2} w_{ij|l}^{|l} + 2k \frac{N^2}{a^2} w_{ij} = 0. \quad (2.28)$$

We will show in the next section that if one assumes that the background satisfies separately Einstein classical field equations, one may achieve a considerable simplification of the Hamiltonian which leads to the quantum equations for the tensor perturbations, which in this case are the sole degrees of freedom to be quantized.

2.2 Classical and quantum gravitational waves on a classical FLRW background

We will now assume that the background spacetime and its matter content are well described by classical GR in order to obtain a simpler Hamiltonian than Eq. (2.27), and a simpler quantum dynamics for the gravitational waves.

Varying the action (2.11) with respect to $a(\tau)$, and keeping only the terms up to zeroth order, we obtain the following equation of motion

$$\frac{\dot{N}}{N} \mathcal{H} - \frac{\ddot{a}}{a} - \frac{\mathcal{H}^2}{2} - \frac{kN^2}{2a^2} - \frac{3\ell_{\text{Pl}}^2}{2} p N^2 = 0, \quad (2.29)$$

which is nothing but one of the Friedmann's equations. On the other hand, integrating by parts the term $w_{ij} \dot{w}^{ij}$ in the action (2.11) yields

$$\begin{aligned}
\mathcal{S} = & \frac{1}{6\ell_{\text{Pl}}^2} \int d^4x \gamma^{1/2} a^3 \left[\frac{1}{N} \left(\frac{\dot{N}}{N} \mathcal{H} - \frac{\mathcal{H}^2}{2} - \frac{\ddot{a}}{a} - \frac{3\ell_{\text{Pl}}^2}{2} p N^2 - \frac{N^2 k}{a^2} \right) w_{ij} w^{ij} \right. \\
& \left. + \frac{1}{4N} \dot{w}_{ij} \dot{w}^{ij} - \frac{N}{4a^2} w_{ij|k} w^{ij|k} - \frac{6\mathcal{H}^2}{N} + \frac{6Nk}{a^2} + 6\ell_{\text{Pl}}^2 N p \right], \quad (2.30)
\end{aligned}$$

where a total time derivative term has been omitted. Using Eq. (2.29), one gets

$$\begin{aligned}
\mathcal{S} = & \frac{1}{6\ell_{\text{Pl}}^2} \int d^4x \gamma^{1/2} a^3 \left(-\frac{6\mathcal{H}^2}{N} + 6\frac{Nk}{a^2} + 6\ell_{\text{Pl}}^2 N p \right. \\
& \left. + \frac{1}{4N} \dot{w}_{ij} \dot{w}^{ij} - \frac{N}{4a^2} w_{ij|k} w^{ij|k} - \frac{Nk}{2a^2} w_{ij} w^{ij} \right). \quad (2.31)
\end{aligned}$$

The second order terms correspond to the action for gravitational waves on a FLRW background as presented in Ref. [1] (shown there in the gauge $N = a$),

$$S_{\text{GW}} = \frac{1}{6\ell_{\text{Pl}}^2} \int d^4x \gamma^{1/2} a^3 \left(\frac{1}{4N} \dot{w}_{ij} \dot{w}^{ij} - \frac{N}{4a^2} w_{ij|k} w^{ij|k} - \frac{Nk}{2a^2} w_{ij} w^{ij} \right). \quad (2.32)$$

Calculating the momenta canonically conjugate to the variables a and w_{ij} then yields

$$P_a = -\frac{2\mathcal{H}a^2V}{\ell_{\text{Pl}}^2 N}, \quad (2.33)$$

and

$$\Pi^{ij} = \frac{\gamma^{1/2} a^3 \dot{w}^{ij}}{12\ell_{\text{Pl}}^2 N}. \quad (2.34)$$

Inverting these relations, and making the canonical transformations (2.15) – (2.18) in the fluid sector, as well as the rescaling (2.19) – (2.25), we obtain for the Hamiltonian of the classical FLRW background and gravitational waves,

$$H = N \left[-\frac{P_a^2}{4a} - ka + \frac{P_\tau}{a^{3(\lambda-1)}} + \int d^3x \left(6\frac{\Pi^{ij}\Pi_{ij}}{\gamma^{1/2}a^3} + \frac{1}{24}\gamma^{1/2}aw_{ij|k}w^{ij|k} + \frac{1}{12}\gamma^{1/2}kw_{ij}w^{ij}a \right) \right]. \quad (2.35)$$

This Hamiltonian also yields the correct Einstein equations at zeroth and first order in the perturbations, in particular Eq. (2.28). Besides, and because only tensor perturbations will be quantized in what follows, we shall focus on the gravitational wave part of Hamiltonian (2.25), namely

$$H_{\text{gw}} = \frac{N}{a} H_{\text{gw}}^c = \int d^3x \left(6\frac{\Pi^{ij}\Pi_{ij}}{\gamma^{1/2}a^2} + \frac{1}{24}\gamma^{1/2}a^2w_{ij|k}w^{ij|k} + \frac{1}{12}\gamma^{1/2}kw_{ij}w^{ij}a^2 \right). \quad (2.36)$$

The transformation achieved here already provides an important simplification over the original Hamiltonian (2.27), but, as we shall now see, one can go even further.

As, for the moment, a is not quantized and satisfies the background classical GR equations, one can view it simply as a function of the time parameter present in the Hamiltonian H_{gw} . One can then put H_{gw} in an even simpler and suggestive form by performing the following time dependent canonical transformation

$$w_{ij} = \frac{\sqrt{12}}{a} \mu_{ij}, \quad \text{and} \quad \Pi^{ij} = \frac{1}{\sqrt{12}} \left(a\Pi_{(\mu)}^{ij} - \gamma^{1/2} \dot{a} \mu^{ij} \right), \quad (2.37)$$

whose generating functional is

$$F_2 \left[w_{ij}, \Pi_{(\mu)}^{ij}, \tau \right] = \frac{1}{\sqrt{12}} \int d^3x \left(aw_{ij} \Pi_{(\mu)}^{ij} - \gamma^{1/2} \frac{a\dot{a}w_{ij}w^{ij}}{2\sqrt{12}} \right), \quad (2.38)$$

from which one obtain the transformation itself through the equations

$$\mu_{ij} = \frac{\delta F_2}{\delta \Pi_{(\mu)}^{ij}}, \quad \text{and} \quad \Pi^{ij} = \frac{\delta F_2}{\delta w_{ij}}. \quad (2.39)$$

The new Hamiltonian will be given by

$$\tilde{H}_{\text{gw}} = H_{\text{gw}} + \frac{\partial F_2}{\partial \tau}, \quad (2.40)$$

and it reads

$$\tilde{H}_{\text{gw}} = \frac{N}{a} \tilde{H}_{\text{gw}}^c = \int d^3x \left[\frac{\Pi_{(\mu)}^{ij} \Pi_{(\mu)ij}}{2\gamma^{1/2}} + \frac{1}{2} \gamma^{1/2} \mu_{ij|k} \mu^{ij|k} + \gamma^{1/2} \left(k - \frac{\ddot{a}}{2a} \right) \mu_{ij} \mu^{ij} \right]. \quad (2.41)$$

In the conformal gauge $N = a$ (for which $\tau = \eta$), the Hamiltonian (2.41) yields the following equation for μ :

$$\ddot{\mu}_{ij} - \mu_{ij|l}^{|l} + \left(2k - \frac{\ddot{a}}{a} \right) \mu_{ij} = 0, \quad (2.42)$$

which can be obtained from Eq. (2.28) by making the substitutions $N = a$ and $w_{ij} = \mu_{ij}/(\sqrt{12}a)$ in this relation.

The classical background geometry furnishes the time parameter (in the gauge chosen, the conformal time) on which the wave functional $\psi(\mu_{ij}, \eta)$ evolves. The functional Schrödinger equation it satisfies, namely

$$i \frac{\partial |\psi\rangle}{\partial \eta} = \hat{H}_{\text{gw}}^c |\psi\rangle, \quad (2.43)$$

reads explicitly, in the coordinate representation,

$$i \frac{\partial \psi(\mu_{ij}, \eta)}{\partial \eta} = \int d^3x \left\{ -\frac{1}{2\gamma^{1/2}} \frac{\delta^2}{\delta \mu_{ij} \delta \mu^{ij}} + \gamma^{1/2} \left[\frac{1}{2} \mu_{ij|k} \mu^{ij|k} + \left(k - \frac{\ddot{a}}{2a} \right) \mu_{ij} \mu^{ij} \right] \right\} \\ \times \psi(\mu_{ij}, \eta) = 0. \quad (2.44)$$

One can also show that quantization in the gauge $N = a$ of gravitational waves using the Hamiltonian H_{gw}^c defined in Eq. (2.36), through the Schrödinger equation

$$i \frac{\partial |\varphi\rangle}{\partial \eta} = \hat{H}_{\text{gw}}^c |\varphi\rangle, \quad (2.45)$$

yields an equivalent quantum theory as the one described by Eq. (2.44). In fact, there is a time dependent quantum canonical transformation mapping the two theories, generated by the unitary operator (for a good review on this subject, see Ref. [9]),

$$U = \exp \left\{ i \left[\int d^3x \gamma^{1/2} \frac{\dot{a} w_{ij} w^{ij}}{2a} \right] \right\} \exp \left\{ i \left[\int d^3x \left(\frac{w_{ij} \Pi^{ij} + \Pi^{ij} w_{ij}}{2} \right) \ln \left(\frac{\sqrt{12}}{a} \right) \right] \right\}. \quad (2.46)$$

Under the map between self-adjoint operators

$$\hat{\hat{A}} = U \hat{A} U^{-1}, \quad (2.47)$$

one can obtain the operator version of Eq. (2.37), making use in the derivation, of the Baker – Campbell – Hausdorff formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots \quad (2.48)$$

The operator $\hat{\hat{H}}_{\text{gw}}^c$ is obtained from \hat{H}_{gw}^c through the usual formula when U is time dependent, namely

$$\hat{\hat{H}}_{\text{gw}}^c = U \hat{H}_{\text{gw}}^c U^{-1} + i \frac{\partial U}{\partial \eta} U^{-1}. \quad (2.49)$$

One can show that if $|\varphi\rangle$ is a solution of Eq. (2.45), then $|\psi\rangle = U|\varphi\rangle$ is a solution of Eq. (2.43) and all probability amplitudes of the two theories are equal as long as U is unitary (the operators in the exponentials are self-adjoint): $\langle\varphi_1|\hat{A}|\varphi_2\rangle = \langle\psi_1|U\hat{A}U^{-1}|\psi_2\rangle = \langle\psi_1|\hat{\hat{A}}|\psi_2\rangle$.

The Schrödinger equation (2.44) is evidently an enormous simplification, made in two steps, over the one that would have been obtained if one had stuck to the Hamiltonian (2.27), with a quite simple interpretation: it describes a scalar field with time dependent mass given by $-\ddot{a}/a$. This time dependence is responsible for the creation and/or annihilation of tensor modes due to the pump field a governing the dynamics of the cosmological background.

The consequences of Eq. (2.44) for tensor perturbations in various classical background models have been extensively discussed in the literature (see e.g. Refs. [1, 2, 13]). We will now investigate the situation where the background is also quantized.

3. Quantum Gravitational Waves on a Quantum FLRW Background

The two Hamiltonians (2.27) and (2.35) are completely equivalent at the classical level, and (2.35) can be obtained from (2.27) if one assumes that the background satisfies classical GR. However, if the background is also quantized, that is, when a becomes an operator, one can obviously no longer use this method to obtain (2.35) from (2.27). Indeed, in Eq. (2.35) a is assumed to be that prescribed function of time (not a canonical variable) which satisfies the classical equations of motion. This is incompatible with a viewed as a quantum operator, coming from the quantization of the canonical variable a . Hence, one may question the use of equation (2.35) as a valid Hamiltonian when the background is also quantized.

It is our goal in this section to show that equation (2.35), with a viewed as a *canonical variable* and *not* as a *prescribed function of time*, can indeed be obtained

from (2.27) *without ever making any assumption concerning the dynamics of the background*, and that the simpler Hamiltonian (2.35) can be used in the canonical quantization of the whole system. This equivalence is also proved at the quantum level. Indeed, simplifying the Hamiltonian constraint is very important at the quantum level since the Dirac canonical quantization procedure for constrained systems is obtained by imposing that the physical states $\Psi(a, w_{ij}, t)$ are annihilated by the operator version of H_0 , i.e. , $\hat{H}_0\Psi = 0$. Using the operator version of Eq. (2.35) happens to be much simpler than that arising from (2.27), which is rather complicated and also suffers from many factor ordering ambiguities.

3.1 The Hamiltonian point of view: classical and quantum canonical transformations

The difference between actions (2.11) and (2.30) is a total time derivative given by,

$$\Delta\mathcal{S} = - \int d^4x \frac{d}{dt} \left(\gamma^{1/2} \frac{\dot{a} a^2 w_{ij} w^{ij}}{6\ell_{\text{Pl}}^2 N} \right) = \int d^4x \frac{d}{dt} \left(\gamma^{1/2} \frac{P_a a w_{ij} w^{ij}}{12\ell_{\text{Pl}}^2} \right), \quad (3.1)$$

which suggests that, after making the redefinitions (2.19) – (2.25), a canonical transformation generated by

$$G = a\tilde{P}_a - \int d^3x \tilde{w}_{ij} \Pi^{ij} + \int d^3x \gamma^{1/2} \frac{\tilde{P}_a a \tilde{w}_{ij} \tilde{w}^{ij}}{12}, \quad (3.2)$$

will transform (2.27) into (2.35) (the first two terms yielding the identity part of the transformation). Indeed, through the relations [here, the tilde variables refer to Hamiltonian (2.27)]

$$\tilde{a} = \frac{\partial G}{\partial \tilde{P}_a}, \quad (3.3)$$

$$P_a = \frac{\partial G}{\partial a}, \quad (3.4)$$

$$w_{ij} = -\frac{\delta G}{\delta \Pi^{ij}}, \quad (3.5)$$

$$\Pi^{ij} = -\frac{\delta G}{\delta \tilde{w}_{ij}}, \quad (3.6)$$

we obtain, up to second order in the perturbation

$$\tilde{a} = a \left(1 + \frac{Q}{12} \right), \quad (3.7)$$

$$\tilde{P}_a = P_a \left(1 - \frac{Q}{12} \right), \quad (3.8)$$

$$\tilde{w}_{ij} = w_{ij}, \quad (3.9)$$

$$\tilde{\Pi}^{ij} = \Pi^{ij} - \frac{1}{6} \gamma^{1/2} a P_a w^{ij}, \quad (3.10)$$

where we have defined

$$Q \equiv \int d^3x \gamma^{1/2} w_{ij} w^{ij}. \quad (3.11)$$

Note that the equation (3.10) is the same as Eq. (2.13). One can easily check that Eqs. (3.7) to (3.10) are canonical transformations (up to second order) which transforms (2.27) into (2.35) (also up to second order),

$$H = N \left[-\frac{P_a^2}{4a} - ka + \frac{P_r}{a^{3(\lambda-1)}} + \int d^3x \left(6 \frac{\Pi^{ij} \Pi_{ij}}{\gamma^{1/2} a^3} + \frac{1}{24} \gamma^{1/2} a w_{ij|k} w^{ij|k} + \frac{1}{12} \gamma^{1/2} k w_{ij} w^{ij} a \right) \right]. \quad (3.12)$$

Eq. (3.12) has the same simple form as Eq. (2.35), but here a and P_a are canonical variables like w_{ij} and Π^{ij} ready to be quantized.

Quantum mechanically, the above canonical transformations are generated by the unitary operator

$$U = \exp(iG_q) \equiv \exp\left(\frac{i}{12} \hat{\beta}_a \hat{Q}\right), \quad (3.13)$$

where $\hat{\beta}_a \equiv (\hat{P}_a \hat{a} + \hat{a} \hat{P}_a)/2$ and $\hat{Q} \equiv \int d^3x \gamma^{1/2} \hat{w}_{ij} \hat{w}^{ij}$ are the self-adjoint operators associated with the corresponding classical variables.

In a way completely analogous to that of the previous section, we perform the map $\hat{A} \mapsto \hat{\hat{A}} = U \hat{A} U^{-1}$ for all operators \hat{A} in such a way as to obtain the operator version of Eq. (3.10), making, here again, use of relation (2.48). Note that the map generated by (3.13) yields $\hat{\hat{a}} = \hat{a} \exp(Q/12)$ and $\hat{\hat{P}}_a = \hat{P}_a \exp(-Q/12)$, which reduces to the operator version of Eq. (3.10) up to second order, with the factor ordering for the transformation of $\hat{\Pi}^{ij}$ given by $\hat{\hat{\Pi}}^{ij} = \hat{\Pi}^{ij} - \gamma^{1/2} \hat{\beta}_a \hat{w}^{ij}/6$.

The transformation of the operator version of (2.27) into the operator version of (2.35) depends on the factor ordering of the operator version of (2.27). For instance, for the ordering

$$\begin{aligned} \hat{H}_0 = & -\frac{\hat{\beta}_a^2}{4\hat{a}^3} - k\hat{a} + \frac{\hat{P}_r}{\hat{a}^{3(\lambda-1)}} \left[1 + \frac{(\lambda-1)}{4} \int d^3x \gamma^{1/2} \hat{w}_{ij} \hat{w}^{ij} \right] + \frac{5\hat{\beta}_a^2}{48\hat{a}^3} \int d^3x \gamma^{1/2} \hat{w}_{ij} \hat{w}^{ij} \\ & + \int d^3x \left[\frac{6\hat{\Pi}_{ij} \hat{\Pi}^{ij}}{\hat{a}^3 \gamma^{1/2}} + \frac{\hat{\beta}_a}{\hat{a}^3} \left(\hat{w}_{ij} \hat{\Pi}^{ij} + \hat{\Pi}^{ij} \hat{w}_{ij} \right) \right] \\ & + \int d^3x \gamma^{1/2} \hat{a} \left(\frac{\hat{w}^{ij|k} \hat{w}_{ij|k}}{24} + \frac{k}{6} \hat{w}_{ij} \hat{w}^{ij} \right), \end{aligned} \quad (3.14)$$

one obtains after this quantum canonical transformation

$$\hat{H}_0 = -\frac{\hat{\beta}_a^2}{4\hat{a}^3} - k\hat{a} + \frac{\hat{P}_r}{\hat{a}^{3(\lambda-1)}} + \int d^3x \left(6 \frac{\hat{\hat{\Pi}}^{ij} \hat{\hat{\Pi}}_{ij}}{\gamma^{1/2} \hat{a}^3} + \frac{1}{24} \gamma^{1/2} a \hat{w}_{ij|k} \hat{w}^{ij|k} + \frac{1}{12} \gamma^{1/2} k \hat{w}_{ij} \hat{w}^{ij} a \right), \quad (3.15)$$

where the ordering of $\hat{\beta}_a^2/\hat{a}^3$ must be the same in both Hamiltonians.

Another possibility is

$$\begin{aligned}\hat{H}_0 = & \left\{ -\frac{1}{4\hat{a}}\hat{P}_a^2 - k\hat{a} + \frac{\hat{P}_r}{\hat{a}^{3(\lambda-1)}} \left[1 + \frac{(\lambda-1)}{4} \int d^3x \gamma^{1/2} \hat{w}_{ij} \hat{w}^{ij} \right] \right. \\ & + \int d^3x \gamma^{1/2} \hat{w}_{ij} \hat{w}^{ij} \left[\frac{1}{24} \left(\hat{a} \hat{P}_a \hat{P}_a \frac{1}{\hat{a}^3} + \hat{a} \hat{P}_a^2 \frac{1}{\hat{a}^2} + \hat{P}_a \hat{a}^2 \hat{P}_a \frac{1}{\hat{a}^3} + \hat{P}_a \hat{a} \hat{P}_a \frac{1}{\hat{a}^2} \right) \right. \\ & \quad \left. \left. - \frac{1}{16\hat{a}} \hat{P}_a^2 \right] \right. \\ & \left. + \int d^3x \left[\frac{6\hat{\Pi}_{ij}\hat{\Pi}^{ij}}{\hat{a}^3\gamma^{1/2}} + \hat{\beta}_a (\hat{w}_{ij}\hat{\Pi}^{ij} + \hat{\Pi}^{ij}\hat{w}_{ij}) \frac{1}{\hat{a}^3} + \gamma^{1/2}\hat{a} \left(\frac{\hat{w}^{ij|k}\hat{w}_{ij|k}}{24} + \frac{k}{6} \hat{w}_{ij}\hat{w}^{ij} \right) \right] \right\},\end{aligned}\quad (3.16)$$

going to

$$\begin{aligned}\hat{H}_0 = & \left[-\frac{1}{4\hat{a}}\hat{P}_a^2 - k\hat{a} + \frac{\hat{P}_r}{\hat{a}^{3(\lambda-1)}} \right. \\ & \left. + \int d^3x \left(6\frac{\hat{\Pi}^{ij}\hat{\Pi}_{ij}}{\gamma^{1/2}\hat{a}^3} + \frac{1}{24}\gamma^{1/2}\hat{a}\hat{w}_{ij|k}\hat{w}^{ij|k} + \frac{1}{12}\gamma^{1/2}k\hat{w}_{ij}\hat{w}^{ij}\hat{a} \right) \right].\end{aligned}\quad (3.17)$$

Quantum evolutions governed by (3.14) and (3.15), or (3.16) and (3.17) are equivalent. It may be noted that the factor ordering of (3.16) and (3.17) yields non self adjoint operators. However, as will see later on, for some fluids they may yield Schrödinger like equations with appropriate self-adjoint reduced Hamiltonian operators, and can therefore be viewed as appropriate quantum representations.

3.2 The Lagrangian point of view

If we start with Eq. (2.11), and make the same integration by parts that led to Eq. (2.30), we obtain an expression that involves \ddot{a} and \dot{N} :

$$\begin{aligned}S = & -\frac{V\dot{a}^2a}{\ell_{\text{Pl}}^2N} + \frac{VN}{\ell_{\text{Pl}}^2} (ka + \ell_{\text{Pl}}^2 pa^3) \\ & + \frac{1}{6\ell_{\text{Pl}}^2} \int d^4x \gamma^{1/2} a^3 \left[\frac{1}{N} \left(\frac{\dot{N}\dot{a}}{Na} - \frac{\dot{a}^2}{2a^2} - \frac{\ddot{a}}{a} - \frac{3\ell_{\text{Pl}}^2}{2} pN^2 - \frac{N^2k}{a^2} \right) w_{ij}w^{ij} \right. \\ & \left. + \frac{1}{4N} \dot{w}_{ij}\dot{w}^{ij} - \frac{N}{4a^2} w_{ij|k}w^{ij|k} \right].\end{aligned}\quad (3.18)$$

In the previous section, it was possible to deal with these terms by using the classical equations of motion. As we do not want to follow this line in this section, we consider another way and make use of the Ostrogradski method [10] to write down a Hamiltonian for a lagrangian that involves second order time derivatives. In this method we define

$$b = \dot{a}, \quad (3.19)$$

and treat the variable b as independent of a . The momenta canonically conjugate to a and b are defined as

$$P_b = \frac{\partial L}{\partial \dot{a}}, \quad \text{and} \quad P_a = \frac{\partial L}{\partial \dot{a}} - \dot{P}_b, \quad (3.20)$$

while the other momenta maintain their usual definitions. They read (the explicit expression for the momentum canonically conjugate to a is not needed in what follows)

$$P_b = -\frac{a^2}{6\ell_{\text{Pl}}^2 N} \int d^3x \gamma^{1/2} w_{ij} w^{ij}, \quad (3.21)$$

$$P_N = \frac{ba^2}{6\ell_{\text{Pl}}^2 N^2} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \quad (3.22)$$

and

$$\Pi^{ij} = \frac{a^3 \gamma^{1/2} \dot{w}^{ij}}{12\ell_{\text{Pl}}^2 N}. \quad (3.23)$$

Equations (3.21) and (3.22) are constraints, which must be added to the Hamiltonian via Lagrange multipliers α and β . In order to simplify the treatment of the constraints, we make the following canonical transformations:

$$\tilde{P}_N = P_N + \frac{bP_b}{N}, \quad (3.24)$$

$$\tilde{N} = N, \quad (3.25)$$

$$\tilde{P}_b = NP_b, \quad (3.26)$$

$$\tilde{b} = \frac{b}{N}, \quad (3.27)$$

and redefine $\tilde{\alpha} \equiv \alpha/N$. After making rescalings as in Eqs.(2.19) – (2.25) and omitting the tilde, the Hamiltonian reads

$$\begin{aligned} H &\equiv NH_0 + \alpha\phi_1 + \beta P_N \\ &= N \left\{ P_a b + b^2 a - ka + \frac{P_T}{a^{3(\lambda-1)}} \left(1 - \frac{1}{4} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \right)^{1-\lambda} \right. \\ &\quad + \frac{b^2 a}{12} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \\ &\quad + \int d^3x \left[\gamma^{1/2} \left(\frac{aw_{ij|k} w^{ij|k}}{24} + \frac{kaw_{ij} w^{ij}}{6} \right) + 6 \frac{\Pi^{ij} \Pi_{ij}}{a^3 \gamma^{1/2}} \right] \Big\} \\ &\quad + \alpha \left(P_b + \frac{a^2}{6} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \right) + \beta P_N. \end{aligned} \quad (3.28)$$

Demanding the conservation of these constraints, that is, the vanishing of their Poisson Brackets with the Hamiltonian (3.28), we obtain the following secondary constraints

$$\phi_2 \equiv -2ba + \frac{ba}{6} \int d^3x \gamma^{1/2} w_{ij} w^{ij} - P_a + \frac{4}{a} \int d^3x w_{ij} \Pi^{ij} \approx 0, \quad (3.29)$$

and

$$H_0 \approx 0. \quad (3.30)$$

Conservation of the secondary constraints $\phi_2 \approx 0$ then fixes the value of α through

$$\alpha = -\frac{\{\phi_2, H_0\}}{\{\phi_2, \phi_1\}}, \quad (3.31)$$

where $\{, \}$ stands for Poisson brackets.

It is straightforward to check that P_N and H_0 are first class constraints, *i.e.* they have vanishing Poisson Brackets with all other constraints, while ϕ_1 and ϕ_2 are second class constraints, which is the reason why the unambiguous determination of α is possible. The way to deal with these second class constraints is to work in the formalism of the Dirac brackets. They are defined as

$$\{A, B\}^D = \{A, B\} - \{A, \phi_i\}(C^{-1})^{ij}\{\phi_j, B\}, \quad (3.32)$$

where $(C^{-1})^{ij}$ is the inverse of the antisymmetric matrix $C_{ij} \equiv \{\phi_i, \phi_j\}$. In our case

$$C_{12} = -C_{21} = 2a \left(1 + \frac{5}{12} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \right). \quad (3.33)$$

By definition, the Dirac brackets of a second class constraint with any phase space function is zero, so we have $\{A, \phi_i\}^D = 0$, and one can view the second class constraints as strong equalities. Hence, from Eq. (3.29) one can write (always up to second order in the perturbations)

$$b = -\frac{P_a}{2a} + \frac{2}{a^2} \int d^3x w_{ij} \Pi^{ij} - \frac{P_a}{24a} \int d^3x \gamma^{1/2} w_{ij} w^{ij}, \quad (3.34)$$

and substitute it into the Hamiltonian (3.28), yielding

$$\begin{aligned} H = N \left\{ -\frac{P_a^2}{4a} - ka + \frac{P_T}{a^{3(\lambda-1)}} + \frac{1}{12} \left[\frac{P_a^2}{4a} + 3(\lambda-1) \frac{P_T}{a^{3(\lambda-1)}} + 2ka \right] \int d^3x \gamma^{1/2} w_{ij} w^{ij} \right. \\ \left. + \int d^3x \left(\frac{\gamma^{1/2} a}{24} w_{ij|k} w^{ij|k} + 6 \frac{\Pi^{ij} \Pi_{ij}}{a^3 \gamma^{1/2}} \right) \right\}. \end{aligned} \quad (3.35)$$

Substituting Eq. (3.33) in Eq. (3.32), we get the following Dirac Brackets between

the dynamical variables (up to second order in the perturbations):

$$\{a, \Pi^{ij}(x)\}^D = \frac{1}{6}aw^{ij}(x), \quad (3.36)$$

$$\{a, w_{ij}(x)\}^D = 0, \quad (3.37)$$

$$\{a, P_a\}^D = 1 + \frac{Q}{6}, \quad (3.38)$$

$$\{a, P_T\}^D = 0, \quad (3.39)$$

$$\{a, T\}^D = 0, \quad (3.40)$$

$$\{P_a, \Pi^{ij}(x)\}^D = \frac{1}{6}\gamma^{1/2}P_aw^{ij}(x), \quad (3.41)$$

$$\{P_a, w_{ij}(x)\}^D = 0, \quad (3.42)$$

$$\{P_a, P_T\}^D = 0, \quad (3.43)$$

$$\{P_a, T\}^D = 0, \quad (3.44)$$

$$\{T, w_{ij}\}^D = 0, \quad (3.45)$$

$$\{T, \Pi^{ij}(x)\}^D = 0, \quad (3.46)$$

$$\{T, P_T\}^D = 1, \quad (3.47)$$

$$\{w_{kl}(x), \Pi^{ij}(x')\}^D = \delta_{kl}^{ij}\delta^3(x - x') - \frac{2}{3}w_{kl}(x)w^{ij}(x'), \quad (3.48)$$

where we have used the definition (3.11) for the quantity Q .

The time evolution of any phase space function is now given by

$$\dot{A} = \{A, H\}^D. \quad (3.49)$$

Hence we will define phase space functions which have canonical Dirac brackets among themselves, namely

$$\{T_{(c)}, P_{(c)T}\}^D = 1, \quad \{a_{(c)}, P_{(c)a}\}^D = 1, \quad (3.50)$$

and

$$\{w_{(c)kl}(x), \Pi_{(c)}^{ij}(x')\}^D = \delta_{kl}^{ij}\delta^3(x - x'), \quad (3.51)$$

the others vanishing. The different new canonical functions are

$$a_{(c)} = a \left(1 - \frac{Q}{12}\right), \quad (3.52)$$

$$P_{(c)a} = P_a \left(1 - \frac{Q}{12}\right), \quad (3.53)$$

$$w_{(c)ij} = w_{ij} \left(1 + \frac{Q}{3}\right). \quad (3.54)$$

If we now express the Hamiltonian (3.35) in terms of these canonical functions, we obtain, omitting the index (c),

$$H = NH_0 = N \left[-\frac{P_a^2}{4a} - ka + \frac{P_T}{a^{3(\lambda-1)}} + \int d^3x \left(6\frac{\Pi^{ij}\Pi_{ij}}{\gamma^{1/2}a^3} + \frac{1}{24}\gamma^{1/2}aw_{ij|k}w^{ij|k} \right) \right]$$

$$+ \frac{1}{12} \gamma^{1/2} k w_{ij} w^{ij} a \Big) \Big], \quad (3.55)$$

which is exactly Hamiltonian (2.35) of the preceding section, also obtained in the previous subsection through the canonical transformations (3.10). Note that at the level of the equations of motion, which are first order in the perturbations, there is no difference between variables with or without the index (c). With the time evolution law (3.49), and remembering that (3.55) is expressed in terms of phase space functions with canonical Dirac brackets relations, it is obvious that (3.55) generates the classical GR equations for this system. Hence, without using any background equations of motion, we were able to obtain the simpler Hamiltonian (3.55), which is equivalent to (2.35), from the more complicated form (2.27), as we did in the last subsection using canonical transformations directly in (2.27).

3.3 The functional Schrödinger equation

As we are here also quantizing the background, the quantization procedure is now to impose $\hat{H}_0 \Psi(a, w_{ij}) = 0$, with \hat{H}_0 being the operator version of Eq. (3.55), where the operator algebra comes from $[,] = i\hbar\{, \}$ in the Hamiltonian point of view, or $[,] = i\hbar\{, \}^D$ in the Lagrangian case, which of course turn to be the same. Evidently, \hat{H}_0 stemming from Eq. (3.55) is much simpler than that coming from Eq. (2.27). We will from now on focus our attention on Eq. (3.55).

The Wheeler-DeWitt equation in this case, with the particular factor ordering of Eq. (3.17) for \hat{H}_0 , reads

$$\left\{ \frac{1}{4a} \frac{\partial^2}{\partial a^2} - ka - i \frac{1}{a^{3(\lambda-1)}} \frac{\partial}{\partial T} + \int d^3x \left[-6 \frac{1}{a^3 \gamma^{1/2}} \frac{\delta^2}{\delta w_{ij} \delta w^{ij}} + a \left(\gamma^{1/2} \frac{w_{ij|k} w^{ij|k}}{24} + k \frac{w_{ij} w^{ij}}{12} \right) \right] \right\} \Psi = 0. \quad (3.56)$$

When the matter fields are described by a perfect fluid, there appears a first order derivative with respect to the variable describing the fluid, which enforces us to interpret it as the time variable on which the wave functional evolves. This is a particular feature of the fluid description, not present when the matter fields are described by, e.g., a scalar field. Choosing T as the time variable is equivalent to impose de time gauge $N = a^{3(\lambda-1)}$. Eq. (3.56) then reads

$$\begin{aligned} i \frac{\partial \Psi}{\partial T} &= \hat{H}_{\text{red}} \Psi \\ &:= \left\{ \frac{a^{3\lambda-4}}{4} \frac{\partial^2}{\partial a^2} - ka^{3\lambda-2} + \int d^3x \left[-6 \frac{a^{3(\lambda-2)}}{\gamma^{1/2}} \frac{\delta^2}{\delta w_{ij} \delta w^{ij}} + a^{3\lambda-2} \left(\gamma^{1/2} \frac{w_{ij|k} w^{ij|k}}{24} + k \frac{w_{ij} w^{ij}}{12} \right) \right] \right\} \Psi. \end{aligned} \quad (3.57)$$

4. Further Developments

In this section we begin by further simplifying the functional Schrödinger equation (3.57) with the use of the Bohm-de Broglie interpretation of quantum mechanics, and then we present, for the sake of completeness, the scalar field case.

4.1 Further simplification of the wave functional equation using the Bohm-de Broglie interpretation

Note that Eq. (3.57), although simpler than the one that would have been obtained through a quantization procedure using Hamiltonian (2.27), has not yet its gravitational wave part in the simplest form (2.44). In fact, Eq. (2.44) was obtained in the previous section through the time dependent quantum canonical transformation generated by Eq. (2.46), which transformed the operator version of Eq. (2.36) into the operator version of Eq. (2.41). This was done under the assumption that $a(T)$ was a given function of the time parameter. However, in a non ontological interpretation of quantum mechanics, the function $a(T)$ does not make sense when a is also quantized since trajectories do not exist in this case. Hence, Eq. (3.57) is the ultimate form that can be achieved in such interpretations.

On the other hand, if one uses an ontological interpretation of quantum mechanics like the one suggested by de Broglie and Bohm [14], and makes the separation ansatz for the wave functional

$$\Psi[a, w_{ij}, T] = \varphi(a, T)\psi[a, w_{ij}, T],$$

with

$$\psi[a, w_{ij}, T] = \psi_1[w_{ij}, T] \int da \varphi^{-2}(a, T) + \psi_2[w_{ij}, T],$$

then Eq. (3.57) can be split into two, yielding

$$i \frac{\partial \varphi}{\partial T} = \frac{a^{3\lambda-4}}{4} \frac{\partial^2 \varphi}{\partial a^2} - k a^{3\lambda-2} \varphi, \quad (4.1)$$

and

$$i \frac{\partial \psi}{\partial T} = \int d^3x \left[-6 \frac{a^{3(\lambda-2)}}{\gamma^{1/2}} \frac{\delta^2}{\delta w_{ij} \delta w^{ij}} + a^{3\lambda-2} \left(\gamma^{1/2} \frac{w_{ij|k} w^{ij|k}}{24} + k \frac{w_{ij} w^{ij}}{12} \right) \right] \psi. \quad (4.2)$$

Using the Bohm interpretation, Eq. (4.1) can now be solved as in Ref. [5, 6, 7], yielding a Bohmian quantum trajectory $a(T)$, which in turn can be used in Eq. (4.2). Indeed, since one can view $a(T)$ as a function of T , it is possible to apply the canonical transformations (2.37) generated by Eq. (2.38) in the same way as in the previous section. With these transformations, Eq. (4.2) reads

$$i \frac{\partial \psi}{\partial T} = \int d^3x \left[-\frac{a^{3\lambda-4}}{2\gamma^{1/2}} \frac{\delta^2}{\delta \mu_{ij} \delta \mu^{ij}} + a^{3\lambda-4} \left(\gamma^{1/2} \frac{\mu_{ij|k} \mu^{ij|k}}{2} + k \mu_{ij} \mu^{ij} - \frac{\ddot{a}}{2a} \mu_{ij} \mu^{ij} \right) \right] \psi, \quad (4.3)$$

Through the redefinition of time $a^{3\lambda-4}dT = d\eta$, we recover Eq. (2.44), namely

$$i \frac{\partial \psi(\mu_{ij}, \eta)}{\partial \eta} = \int d^3x \left\{ -\frac{1}{2\gamma^{1/2}} \frac{\delta^2}{\delta \mu_{ij} \delta \mu^{ij}} + \gamma^{1/2} \left[\frac{1}{2} \mu_{ij|k} \mu^{ij|k} + \left(k - \frac{\ddot{a}}{2a} \right) \mu_{ij} \mu^{ij} \right] \right\} \times \psi(\mu_{ij}, \eta). \quad (4.4)$$

This is the most simple form of the Schrödinger equation which governs tensor perturbations for fluid matter source. In this way, we can proceed with the usual analysis, with the quantum Bohmian solution $a(\eta)$ coming from Eq. (4.1) acting as the new pump field.

If the fluid is radiation, which is one of the best phenomenological approach for the primordial Universe, then $\lambda = \frac{4}{3}$, and there is no factor ordering ambiguity neither in the operator version of Eq. (3.55) (in this case conformal time is obtained through setting $N = a$) nor in Eq. (4.1). Furthermore, in a restricted Hilbert space, the operator read in the right-hand-side of Eq. (4.1) can be made self adjoint [15].

4.2 The scalar field case

In order to be complete, we end this section by a rapid examination of the specific case of a scalar field to exhibit specifically the differences with the fluid case.

In the case of a minimally coupled scalar field, the scalar field action which must be added to the gravitational action (2.8) reads

$$S_\phi = \int d^4x \sqrt{-g} \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] = \int d^4x \gamma^{1/2} a^3 \left\{ \frac{\dot{\phi}^2}{2N} - NV(\phi) - \frac{1}{4} w_{ij} w^{ij} \left[\frac{\dot{\phi}^2}{2N} - NV(\phi) \right] \right\}, \quad (4.5)$$

where we have used the form (2.4) of the perturbed ADM metric and expanded the action to second order in the perturbations as before.

Calculating the Hamiltonian in the standard way, and performing redefinitions as in Eqs. (2.25), we obtain

$$\begin{aligned} H &\equiv NH_0 \\ &= N \left\{ -\frac{P_a^2}{4a} - ka + \frac{\Pi_\phi^2}{2a^3} + a^3 V(\phi) \left(1 - \frac{1}{4} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \right) \right. \\ &\quad \left. + \frac{5P_a^2}{48a} \int d^3x \gamma^{1/2} w_{ij} w^{ij} \right. \\ &\quad \left. + \int d^3x \left[\frac{6\tilde{\Pi}_{ij}\tilde{\Pi}^{ij}}{a^3\gamma^{1/2}} + 2\frac{P_a w_{ij}\tilde{\Pi}^{ij}}{a^2} + \gamma^{1/2} a \left(\frac{w^{ij|k} w_{ij|k}}{24} + \frac{k}{6} w_{ij} w^{ij} \right) \right] \right\}, \quad (4.6) \end{aligned}$$

where Π_ϕ is the momentum canonically conjugated to ϕ ; this is exactly the Hamiltonian obtained in Ref. [8] for tensor perturbations.

After performing the canonical transformations of subsection 3.1, we obtain finally

$$H = N \left[-\frac{P_a^2}{4a} - ka + \frac{\Pi_\phi^2}{2a^3} + a^3 V(\phi) + \int d^3x \left(6 \frac{\Pi^{ij} \Pi_{ij}}{\gamma^{1/2} a^3} + \frac{1}{24} \gamma^{1/2} a w_{ij|k} w^{ij|k} + \frac{1}{12} \gamma^{1/2} k w_{ij} w^{ij} a \right) \right], \quad (4.7)$$

which is, as before, a great simplification over Eq. (4.6). Note, however, that, in the case of the scalar field, there is no linear momentum which can be obviously chosen as a time derivative as in the perfect fluid case [see Eq. (2.35) for comparison]. To obtain equations like (4.1) and (4.2), and then (4.3), one must make a choice of time, which is rather arbitrary [16], and problematic [17] in this situation. Anyway, at the semiclassical level it is certain that Eq. (4.3) can be obtained adopting procedures similar to what was done in Ref. [8] for the quantum equation coming from Hamiltonian (4.7).

5. Conclusion

Considering the case for which the background is quantized, we have obtained the form of the quantum equations governing tensor perturbations. The result can be made as simple as when the background satisfies Einstein classical field equations. This we achieved in two successive steps.

The first step is general and the simplification is obtained through either classical and quantum canonical transformations in the Hamiltonian point of view, or through the elimination of a total time derivative and the subsequent Ostrogradski treatment of second time derivatives which appear in the lagrangian formalism.

When the matter source is described by a perfect fluid and a notion of time appears naturally, the second step demands, in order to be implemented, using the Bohm-de Broglie interpretation of quantum mechanics. In this framework, a real quantum trajectory $a(t)$ in minisuperspace does exist, and a further quantum canonical transformation can be made in order to put the quantum equations exactly in the same form that appears in most previous works dealing with the evolution of cosmological tensor perturbations in classical backgrounds. The only difference is that the time dependent effective mass is no longer calculated in terms of the classical background solution for $a(t)$, but in terms of the quantum Bohmian solution $a(t)$, which have been obtained in all fluid cases [5, 7]. Hence, in the Bohmian approach, the quantum background effects turn out to be very simple to calculate, much easier than in other, non ontological, interpretations of quantum mechanics.

It is interesting to ask the question as to whether the Bohmian approach, which turns out to be also the simplest for this situation, will provide solutions for the observable predictions that differ from the other interpretations of quantum mechanics

applicable to quantum cosmology [18]. It could be conjectured that self-consistency of quantum principles would imply that the outcome of the calculation should eventually not depend of the interpretation, so that using the Bohm trajectories would then be reducible merely to a clever computational trick. However, unless the actual calculation is performed, the opposite can also be conjectured, namely that there could exist different predictions in the quantum cosmological framework that could allow, in principle and thus possibly observationally, to discriminate between various alternatives. The case of matter being described by a scalar field, as was briefly discussed here, needs further elaboration: the problem here lies in the absence of a natural time. Whatever the answer to the abovementioned problems, it is clear that the framework of Bohm interpretation seems adequate in order to go further.

Whether similar calculations and simplification can be implemented in the scalar [19] and vector perturbation cases is rather doubtful and yet under study [20]. We guess that the simplification obtained here, completely independent of the background equations, is a feature of tensor perturbations, and tensor only, therefore most presumably not to be generalized to scalar and vector perturbations. In fact, we believe this is related to the fact that only tensor perturbations do not have any linear perturbation terms present in the action, as such terms happen to vanish identically. This is not the case, e.g., of the action of scalar perturbations, which does contain such linear terms, with coefficients that vanish provided the background functions satisfy the background equations of motion, as it should. Also, the form of the equations of motion governing the dynamics of tensor perturbations is completely independent of the matter source of the gravitational field, contrary, e.g., to the scalar perturbations case.

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